
More results on the number of zeros of multiplicity at least r

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Abstract: We consider multivariate polynomials and investigate how many zeros of multiplicity at least r they can have over a Cartesian product of finite subsets of a field. Here r is any prescribed positive integer and the definition of multiplicity that we use is the one related to Hasse derivatives. As a generalization of material in [2, 5] a general version of the Schwartz-Zippel was presented in [8] which from the leading monomial – with respect to a lexicographic ordering – estimates the sum of zeros when counted with multiplicity. The corresponding corollary on the number of zeros of multiplicity at least r is in general not sharp and therefore in [8] a recursively defined function D was introduced using which one can derive improved information. The recursive function being rather complicated, the only known closed formula consequences of it are for the case of two variables [8]. In the present paper we derive closed formula consequences for arbitrary many variables, but for the powers in the leading monomial being not too large. Our bound can be viewed as a generalization of the footprint bound [10, 6] – the classical footprint bound taking not multiplicity into account.

Keywords: Footprint bound, multiplicity, multivariate polynomial, Schwartz-Zippel bound, zeros of polynomial

MSC classifications: Primary: 12Y05. Secondary: 11T06, 12E05, 13P05, 26C99

1 Introduction

Given a univariate polynomial over an arbitrary field it is an easy task to estimate the number of zeros of multiplicity at least r , for any fixed positive integer r . As is well-known the number of such zeros is less than or equal to the degree of the polynomial divided by r . For multivariate polynomials the situation is much more complicated as these polynomials on the one hand typically have an infinite number of zeros when the field is infinite and on the other hand have only a finite number of zeros when not. A meaningful reformulation of the problem which works independently of the field – and which will be taken in the present paper – is to restrict to point sets that are Cartesian products of finite sets. This of course includes the important case where the point set is $\mathbb{F}_q \times \cdots \times \mathbb{F}_q$, \mathbb{F}_q being the finite field with q elements. Another concern is which definition of multiplicity to use as for multivariate polynomials there are more competing definitions. In the present paper we use the one related to Hasse derivatives (see Definition 2 below).

The interest in studying the outlined problem originally came from applications to Guruswami-Sudan style [9] list decoding algorithms for q -ary Reed-Muller codes, weighted Reed-Muller codes and their likes [11, 12, 1, 2, 8]. The first bound on the number of zeros of prescribed multiplicity was developed by Pellikaan and Wu in [11,

12]. Later Augot and Stepanov improved upon Pellikaan and Wu's bound (see [8, Prop. 13]) by generalizing the Schwartz-Zippel bound to also deal with multiplicity [2]. The proof of this bound was later given by Dvir et al. in [5] where it was used to estimate the size of Kakeya sets over finite fields. The mentioned Schwartz-Zippel bound estimates the sum of zeros when counted with multiplicity. From this, one obtains an easy corollary on the number of zeros of multiplicity r or more. All of the above mentioned bounds are stated in terms of the total degree of the involved polynomials and the point set under consideration is always $\mathbb{F}_q \times \cdots \times \mathbb{F}_q$. In [8, Th. 5] the generalization of the Schwartz-Zippel bound was taken a step further to now work for arbitrary finite point sets $S_1 \times \cdots \times S_m$, $S_i \subseteq \mathbb{F}$, $i = 1, \dots, m$ (where \mathbb{F} is any field) and to take into account the leading monomial with respect to a lexicographic ordering. Again one obtains an easy corollary on the number of zeros of multiplicity at least r [8, Cor. 3]. Whereas the generalized Schwartz-Zippel bound [8, Th. 5] is tight in the sense that we can always find polynomials attaining it (see Proposition 7 below) a similar result does not hold for its corollary [8, Cor. 3]. To address this problem we introduced in [8] a recursively defined function D to estimate the number of zeros of multiplicity at least r . Unfortunately, the function D is quite complicated and only for the case of two variables some simple closed formula upper bounds were derived [8, Prop. 16].

The purpose of the present paper is to establish for the general case of arbitrarily many variables a class of cases in which from D we can derive a simple closed formula expression which is still an improvement to the Schwartz-Zippel bound for zeros of multiplicity at least r ([8, Cor. 3]). The bound that we derive turns out to be a natural generalization of the footprint bound [10, 6] which estimates the number of zeros without taking multiplicity into consideration.

The paper is organized as follows. In Section 2 we start by defining multiplicity and by recalling the general Schwartz-Zippel bound and as a corollary the Schwartz-Zippel bound for zeros of multiplicity at least r . The rest of Section 2 is devoted to a discussion of the method from [8]. In Section 3 we give the new results regarding a simple closed formula upper bound for the case of the coefficients in the leading monomial being small. The concept of being small in general is rather involved and we therefore establish simple sufficient conditions for this to happen.

2 Background

We first recall the concept of Hasse derivatives.

Definition 1 Given $F(X_1, \dots, X_m) \in \mathbb{F}[X_1, \dots, X_m]$ and $\mathbf{k} = (k_1, \dots, k_m) \in \mathbb{N}_0^m$ the \mathbf{k} 'th Hasse derivative of F , denoted by $F^{(\mathbf{k})}(X_1, \dots, X_m)$ is the coefficient of $Z_1^{k_1} \cdots Z_m^{k_m}$ in $F(X_1 + Z_1, \dots, X_m + Z_m) \in \mathbb{F}(X_1, \dots, X_m)[Z_1, \dots, Z_m]$. In other words

$$F(X_1 + Z_1, \dots, X_m + Z_m) = \sum_{\mathbf{k}} F^{(\mathbf{k})}(X_1, \dots, X_m) Z_1^{k_1} \cdots Z_m^{k_m}.$$

Observe that the next definition includes the usual concept of multiplicity for univariate polynomials as a special case.

Definition 2 For $F(X_1, \dots, X_m) \in \mathbb{F}[X_1, \dots, X_m] \setminus \{0\}$ and $\mathbf{a} = (a_1, \dots, a_m) \in \mathbb{F}^m$ we define the multiplicity of F at \mathbf{a} denoted by $\text{mult}(F, \mathbf{a})$ as follows. Let r be an integer such that for every $\mathbf{k} = (k_1, \dots, k_m) \in \mathbb{N}_0^m$ with $k_1 + \cdots + k_m < r$, $F^{(\mathbf{k})}(a_1, \dots, a_m) = 0$ holds, but for some $\mathbf{k} = (k_1, \dots, k_m) \in \mathbb{N}_0^m$ with $k_1 + \cdots + k_m = r$, $F^{(\mathbf{k})}(a_1, \dots, a_m) \neq 0$ holds, then $\text{mult}(F, \mathbf{a}) = r$. If $F = 0$ then we define $\text{mult}(F, \mathbf{a}) = \infty$.

The above definition is the one that is usually given in the literature. For our purpose the below equivalent description shall also prove useful.

Definition 3 Let $F(X_1, \dots, X_m) \in \mathbb{F}[X_1, \dots, X_m] \setminus \{0\}$ and $\mathbf{a} = (a_1, \dots, a_m) \in \mathbb{F}^m$. Consider the ideal

$$J_t = \langle (X_1 - a_1)^{p_1} \cdots (X_m - a_m)^{p_m} \mid p_1 + \cdots + p_m = t \rangle \subseteq \mathbb{F}[X_1, \dots, X_m].$$

We have $\text{mult}(F, \mathbf{a}) = r$ if $F \in J_r \setminus J_{r+1}$. If $F = 0$ we have $\text{mult}(F, \mathbf{a}) = \infty$.

We next state the most general form of the Schwartz-Zippel bound for fields [8, Th. 5]. Here, and in the rest of the paper $S_1, \dots, S_m \subset \mathbb{F}$ are finite subsets of the field \mathbb{F} and we write $s_1 = |S_1|, \dots, s_m = |S_m|$. We note that the below theorem was generalized to arbitrary commutative rings in [3, Th. 7.10] where it was called the generalized Schwartz Theorem.

Theorem 4 Let $F(X_1, \dots, X_m) \in \mathbb{F}[X_1, \dots, X_m]$ be a non-zero polynomial and let $X_1^{i_1} \cdots X_m^{i_m}$ be its leading monomial with respect to a lexicographic ordering \prec_{lex} . Then for any finite sets $S_1, \dots, S_m \subseteq \mathbb{F}$

$$\sum_{\mathbf{a} \in S_1 \times \cdots \times S_m} \text{mult}(F, \mathbf{a}) \leq i_1 s_2 \cdots s_m + s_1 i_2 s_3 \cdots s_m + \cdots + s_1 \cdots s_{m-1} i_m.$$

Turning to the problem of estimating the number of zeros of multiplicity at least r – which is the topic of the present paper – we have the following corollary corresponding to [8, Cor. 3]. We may think of it as the Schwartz-Zippel bound for zeros of multiplicity at least r .

Corollary 5 Let $F(X_1, \dots, X_m) \in \mathbb{F}[X_1, \dots, X_m]$ be a non-zero polynomial and let $X_1^{i_1} \cdots X_m^{i_m}$ be its leading monomial with respect to the lexicographic ordering. Assume $S_1, \dots, S_m \subseteq \mathbb{F}$ are finite sets. Then over $S_1 \times \cdots \times S_m$ the number of zeros of multiplicity at least r is less than or equal to the minimum of

$$(i_1 s_2 \cdots s_m + s_1 i_2 s_3 \cdots s_m + \cdots + s_1 \cdots s_{m-1} i_m) / r$$

and $s_1 \cdots s_m$.

As mentioned in the introduction one obtains better estimates than Corollary 5 by using the recursively defined function D . In particular Corollary 5 is not tight. Before giving the details we pause for a moment to show that on the other hand Theorem 4 is tight (a fact that has not been reported before). For this purpose we shall need the notation

$$S_j = \{\alpha_1^{(j)}, \dots, \alpha_{s_j}^{(j)}\}$$

for $j = 1, \dots, m$, and the below proposition:

Proposition 6 Consider

$$F(X_1, \dots, X_m) = \prod_{u=1}^m \prod_{v=1}^{s_u} (X_u - \alpha_v^{(u)})^{r_v^{(u)}}. \quad (1)$$

The multiplicity of $(\alpha_{j_1}^{(1)}, \dots, \alpha_{j_m}^{(m)})$ in $F(X_1, \dots, X_m)$ equals

$$r_{j_1}^{(1)} + \cdots + r_{j_m}^{(m)}. \quad (2)$$

Proof: Clearly, the multiplicity is greater than or equal to $r = r_{j_1}^{(1)} + \dots + r_{j_m}^{(m)}$. Using Gröbner basis theory we now show that it is not larger. We substitute $\mathcal{X}_i = X_i - \alpha_{j_i}^{(i)}$ for $i = 1, \dots, m$ and observe that by Buchberger's S-pair criteria

$$\mathcal{B} = \{\mathcal{X}_1^{r_1} \dots \mathcal{X}_m^{r_m} \mid r_1 + \dots + r_m = r + 1\}$$

is a Gröbner basis (with respect to any fixed monomial ordering). The support of $F(\mathcal{X}_1, \dots, \mathcal{X}_m)$ contains a monomial of the form $\mathcal{X}_1^{i_1} \dots \mathcal{X}_m^{i_m}$ with $i_1 + \dots + i_m = r$. Therefore the remainder of $F(\mathcal{X}_1, \dots, \mathcal{X}_m)$ modulo \mathcal{B} is non-zero. It is well known that if a polynomial is reduced modulo a Gröbner basis then the remainder is zero if and only if it belongs to the ideal generated by the elements in the basis. \square

We are now ready to show that Theorem 4 is tight.

Proposition 7 *Let $S_1, \dots, S_m \subseteq \mathbb{F}$ be finite sets. If $F(X_1, \dots, X_m) \in \mathbb{F}[X_1, \dots, X_m]$ is a product of univariate linear factors – meaning that it is of the form (1) – then the number of zeros of F counted with multiplicity reaches the generalized Schwartz-Zippel bound (Theorem 4).*

Proof: Consider the polynomial

$$F(X_1, \dots, X_m) = \prod_{u=1}^m \prod_{v=1}^{s_u} (X_u - \alpha_v^{(u)})^{r_v^{(u)}}.$$

Write $i_u = \sum_{v=1}^{s_u} r_v^{(u)}$, $u = 1, \dots, m$. Applying carefully Proposition 6 we obtain

$$\begin{aligned} \sum_{\mathbf{a} \in S_1 \times \dots \times S_m} \text{mult}(F, \mathbf{a}) &= \sum_{t=1}^{s_1} (s_2 \dots s_m) r_t^{(1)} + \dots + \sum_{t=1}^{s_m} (s_1 \dots s_{m-1}) r_t^{(m)} \\ &= i_1 s_2 \dots s_m + \dots + s_1 \dots s_{m-1} i_m \end{aligned}$$

and we are through. \square

We next return to the problem of improving Corollary 5 for which we introduced in [8, Def. 5] the function D .

Definition 8 *Let $r \in \mathbb{N}, i_1, \dots, i_m \in \mathbb{N}_0$. Define*

$$D(i_1, r, s_1) = \min \left\{ \left\lfloor \frac{i_1}{r} \right\rfloor, s_1 \right\}$$

and for $m \geq 2$

$$\begin{aligned} D(i_1, \dots, i_m, r, s_1, \dots, s_m) &= \\ \max_{(u_1, \dots, u_r) \in A(i_m, r, s_m)} &\left\{ (s_m - u_1 - \dots - u_r) D(i_1, \dots, i_{m-1}, r, s_1, \dots, s_{m-1}) \right. \\ &+ u_1 D(i_1, \dots, i_{m-1}, r-1, s_1, \dots, s_{m-1}) + \dots \\ &\left. + u_{r-1} D(i_1, \dots, i_{m-1}, 1, s_1, \dots, s_{m-1}) + u_r s_1 \dots s_{m-1} \right\} \end{aligned}$$

where

$$\begin{aligned} A(i_m, r, s_m) &= \\ \{(u_1, \dots, u_r) \in \mathbb{N}_0^r \mid &u_1 + \dots + u_r \leq s_m \text{ and } u_1 + 2u_2 + \dots + ru_r \leq i_m\}. \end{aligned} \quad (3)$$

Throughout the rest of the paper we shall always assume that $r \in \mathbb{N}$ and that $i_1, \dots, i_m \in \mathbb{N}_0$. The improvement of Corollary 5 was given in [8, Th. 6] as follows:

Table 1 $D(i_1, i_2, 3, 5, 5)$

		i_1														
		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
i_2	0	0	0	0	5	5	5	10	10	10	15	15	15	20	20	20
	1	0	0	1	5	6	6	11	11	12	16	17	17	21	21	21
	2	0	1	2	7	8	9	13	13	14	17	19	19	22	22	22
	3	5	5	5	9	9	10	14	14	16	18	21	21	23	23	23
	4	5	5	6	9	11	13	16	16	18	19	23	23	24	24	24
	5	5	6	7	11	12	14	17	17	20	20					
	6	10	10	10	13	14	17	19	19	21	21					
	7	10	10	11	13	15	18	20	20	22	22					
	8	10	11	12	15	17	21	22	22	23	23					
	9	15	15	15	17	18	22	23	23	24	24					
	10	15	15	16	17	20										
	11	15	16	17	19	21										
	12	20	20	20	21	22										
	13	20	20	21	21	23										
	14	20	21	22	23	24										

Table 2 The Schwartz-Zippel bound (sz) for zeros of multiplicity at least 3

$i_1 + i_2$	0	1	2	3	4	5	6	7	8	9	10	11
sz	0	1	3	5	6	8	10	11	13	15	16	18
$i_1 + i_2$	12	13	14	15	16	17	18					
sz	20	21	23	25	25	25	25					

Theorem 9 For a polynomial $F(X_1, \dots, X_m) \in \mathbb{F}[X_1, \dots, X_m]$ let $X_1^{i_1} \dots X_m^{i_m}$ be its leading monomial with respect to the lexicographic ordering \prec_{lex} with $X_m \prec_{lex} \dots \prec_{lex} X_1$. Then F has at most $D(i_1, \dots, i_m, r, s_1, \dots, s_m)$ zeros of multiplicity at least r in $S_1 \times \dots \times S_m$. The corresponding recursive algorithm produces a number that is at most equal to the number found in Corollary 5 and is at most equal to $s_1 \dots s_m$.

When $\lfloor i_1/s_1 \rfloor + \dots + \lfloor i_m/s_m \rfloor \geq r$ Proposition 6 guarantees the existence of polynomials $F(X_1, \dots, X_m)$ with leading monomial $X_1^{i_1} \dots X_m^{i_m}$ having all elements of $S_1 \times \dots \times S_m$ as zeros of multiplicity at least r . Hence, we only need to apply Theorem 9 to the case $\lfloor i_1/s_1 \rfloor + \dots + \lfloor i_m/s_m \rfloor < r$, and in particular we can assume $i_t < r s_t$.

Example 10 In this example we estimate the number of zeros of multiplicity 3 or more for polynomials in two variables. Both S_1 and S_2 are assumed to be of size 5. From the above discussion, for

$$(i_1, i_2) \in \{(\alpha, \beta) \mid \alpha \geq 15\} \cup \{(\alpha, \beta) \mid \alpha \geq 10 \text{ and } \beta \geq 5\} \\ \cup \{(\alpha, \beta) \mid \alpha \geq 5 \text{ and } \beta \geq 10\} \cup \{(\alpha, \beta) \mid \beta \geq 15\}$$

we have $D(i_1, i_2, 3, 5, 5) = 25$. Table 1 shows information obtained from our algorithm for the remaining possible choices of exponents (i_1, i_2) . Observe, that the table is not symmetric meaning that $D(i_1, i_2, 3, 5, 5)$ does not always equal $D(i_2, i_1, 3, 5, 5)$. The corresponding values of the Schwartz-Zippel bound (Corollary 5) is displayed in Table 2, from which it is clear that indeed the function D can sometimes give a dramatic improvement. For instance $D(3, 11, 3, 5, 5)$ equals 19, but the Schwartz-Zippel bound only gives the estimate 23. Similarly, $D(2, 12)$ equals 20 and the Schwartz-Zippel bound gives 23.

It is easy to establish a lower bound on the maximal number of possible zeros of multiplicity at least $r = 3$ for polynomials with any leading monomial $X_1^{i_1} X_2^{i_2}$. This is done by inspecting polynomials of the form (1). As an example $\prod_{u=1}^4 (X_1 -$

Table 3 Difference between upper and lower bound in Example 10

		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
	i_1															
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	1	0	0	0	0	1	0	1	1	1	1	2	1	1	1	0
	2	0	0	0	2	2	2	3	2	2	2	3	2	2	1	0
	3	0	0	0	0	0	1	1	1	3	1	4	3	2	2	0
	4	0	0	0	0	2	3	3	3	2	2	3	2	2	1	0
	5	0	0	0	2	2	3	2	2	0	0					
	6	0	0	0	0	1	2	3	2	1	0					
i_2	7	0	0	0	0	2	3	3	3	1	0					
	8	0	0	0	2	1	1	2	1	2	0					
	9	0	0	0	0	1	2	2	1	1	0					
	10	0	0	0	0	0										
	11	0	0	0	1	0										
	12	0	0	0	0	0										
	13	0	0	0	0	0										
	14	0	0	0	0	0										

$\alpha_u^{(1)2} \prod_{v=1}^5 (X_2 - \alpha_v^{(2)})$ has 20 zeros of multiplicity (at least) 3. But $D(8, 5, 3, 5, 5) = 20$ and therefore the true value of the maximal number of zeros of multiplicity at least 3 is 20 in this case. In Table 3 we list the difference between $D(i_1, i_2, 3, 5, 5)$ and the lower bound found by using the above method. The large amount of zero's in the table proves that $D(i_1, i_2, 3, 5, 5)$ often equals the true maximal number of zeros of multiplicity at least 3.

In [8, Pro. 16] we derived the following closed formula expression upper bounds for the case of two variables.

Proposition 11 For $k = 1, \dots, r-1$, $D(i_1, i_2, r, s_1, s_2)$ is upper bounded by

$$(C.1) \quad s_2 \frac{i_1}{r} + \frac{i_2}{r} \frac{i_1}{r-k}$$

$$\text{if } (r-k) \frac{r}{r+1} s_1 \leq i_1 < (r-k) s_1 \quad \text{and} \quad 0 \leq i_2 < k s_2$$

$$(C.2) \quad s_2 \frac{i_1}{r} + ((k+1) s_2 - i_2) \left(\frac{i_1}{r-k} - \frac{i_1}{r} \right) + (i_2 - k s_2) \left(s_1 - \frac{i_1}{r} \right)$$

$$\text{if } (r-k) \frac{r}{r+1} s_1 \leq i_1 < (r-k) s_1 \quad \text{and} \quad k s_2 \leq i_2 < (k+1) s_2$$

$$(C.3) \quad s_2 \frac{i_1}{r} + \frac{i_2}{k+1} \left(s_1 - \frac{i_1}{r} \right)$$

$$\text{if } (r-k-1) s_1 \leq i_1 < (r-k) \frac{r}{r+1} s_1 \quad \text{and} \quad 0 \leq i_2 < (k+1) s_2.$$

Finally,

$$(C.4) \quad D(i_1, i_2, r, s_1, s_2) = s_2 \lfloor \frac{i_1}{r} \rfloor + i_2 \left(s_1 - \lfloor \frac{i_1}{r} \rfloor \right)$$

$$\text{if } s_1(r-1) \leq i_1 < s_1 r \quad \text{and} \quad 0 \leq i_2 < s_2.$$

The above numbers are at most equal to $\min\{(i_1 s_2 + s_1 i_2)/r, s_1 s_2\}$.

If in (C.3) of the above proposition we substitute $k = r-1$ then we derive

$$D(i_1, i_2, r, s_1, s_2) \leq s_1 s_2 - \left(s_1 - \frac{i_1}{r} \right) \left(s_2 - \frac{i_2}{r} \right) \quad (4)$$

for $0 \leq i_1 < \frac{r}{r+1} s_1$ and $0 \leq i_2 < r s_2$. Actually, (4) holds under the weaker assumption

$$0 \leq i_1 \leq \frac{r}{r+1} s_1, 0 \leq i_2 < r s_2 \quad (5)$$

which is seen by plugging in the values $k = r-1$ and $i_1 = \frac{r}{r+1} s_1$ into the expressions in (C.1), (C.2) and (4). This is the result that we will generalize to more variables in the next section.

Table 4 $\lfloor 25 - (5 - i_1/3)(5 - i_2/3) \rfloor$

		i_1															
		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	
i_2	0	0	1	3	5	6	8	10	11	13	15	16	18	20	21	23	
	1	1	3	4	6	7	9	11	12	14	<u>15</u>	17	18	<u>20</u>	21	23	
	2	3	4	6	7	9	10	<u>12</u>	13	14	<u>16</u>	<u>17</u>	19	<u>20</u>	22	23	
	3	5	6	7	9	10	11	<u>13</u>	14	<u>15</u>	<u>17</u>	<u>18</u>	<u>19</u>	<u>21</u>	<u>22</u>	23	
	4	6	7	9	10	11	<u>12</u>	<u>14</u>	<u>15</u>	<u>16</u>	<u>17</u>	<u>18</u>	<u>20</u>	<u>21</u>	<u>22</u>	<u>23</u>	
	5	8	9	10	11	12	<u>13</u>	<u>15</u>	<u>16</u>	<u>17</u>	<u>18</u>						
	6	10	11	12	13	14	<u>15</u>	<u>16</u>	<u>17</u>	<u>18</u>	<u>19</u>						
	7	11	12	13	14	15	<u>16</u>	<u>17</u>	<u>17</u>	<u>18</u>	<u>19</u>						
	8	13	14	14	15	<u>16</u>	<u>17</u>	<u>18</u>	<u>18</u>	<u>19</u>	<u>20</u>						
	9	15	15	16	17	<u>17</u>	<u>18</u>	<u>19</u>	<u>19</u>	<u>20</u>	<u>21</u>						
	10	16	17	17	18	<u>18</u>											
	11	18	18	19	19	<u>20</u>											
	12	20	20	20	21	<u>21</u>											
	13	21	21	22	22	<u>22</u>											
	14	23	23	23	23	<u>23</u>											

Example 12 This is a continuation of Example 10 where we investigated $D(i_1, i_2, 3, 5, 5)$. Although condition (5) reads $i_1 \leq 3$ and $i_2 \leq 14$ we print in Table 4 the value of (4) for all possible (i_1, i_2) . The single, as well as double, underlined numbers correspond to entries where the number is strictly smaller than $D(i_1, i_2, 3, 5, 5)$. For such entries (4) certainly doesn't hold true. By inspection, condition (5) seems rather sharp. The double underlined numbers correspond to cases where even, the value is smaller than the lower bounds on the maximal number of zeros, that we established at the end of Example 10. Hence, not only cannot (4) serve as a general upper bound on D , but neither can it serve as a general upper bound on the maximal number of zeros of multiplicity at least r .

3 A closed formula expression when (i_1, \dots, i_m) is small

Having already four different cases of closed formula expressions when $m = 2$ (Proposition 11), the situation gets very complicated for more variables. Assuming, however, that the exponent (i_1, \dots, i_m) in the leading monomial is “small” – a concept that will be formally defined in Definition 15 below – we can give a simple formula which is a generalization of (4) and which is also strongly related to the footprint bound from Gröbner basis theory.

Given a zero dimensional ideal of a multivariate polynomial ring, and a fixed monomial ordering, the well-known footprint bound states that the size of the corresponding variety is at most equal to the number of monomials that can not be found as leading monomial of any polynomial in the ideal (if moreover the ideal is radical, then equality holds). More details on the footprint bound can be found in [4, 10, 6] – in particular see [4, Pro. 4, Sec. 5.3]. We have the following easy corollary.

Corollary 13 Given a polynomial $F(X_1, \dots, X_m) \in \mathbb{F}[X_1, \dots, X_m]$, and a monomial ordering, let $X_1^{i_1} \cdots X_m^{i_m}$ be the leading monomial of F , and assume $i_1 < s_1, \dots, i_m < s_m$. The number of elements in $S_1 \times \cdots \times S_m$ that are zeros of F is at most equal to

$$s_1 \cdots s_m - (s_1 - i_1)(s_2 - i_2) \cdots (s_m - i_m). \quad (6)$$

Proof: The set of zeros of F from $S_1 \times \cdots \times S_m$ equals the variety of the ideal $\langle F, G_1, \dots, G_m \rangle$ where $G_i = \prod_{u=1}^{s_i} (X_i - \alpha_u^{(i)})$. Here, we used the notation introduced

prior to Proposition 6. The above ideal clearly is zero-dimensional. In fact, the monomials that are not leading monomial of any polynomial in the ideal must belong to the set

$$\{X_1^{j_1} \cdots X_m^{j_m} \mid j_1 < s_1, \dots, j_m < s_m, X_1^{j_1} \cdots X_m^{j_m} \text{ is not divisible by } X_1^{i_1} \cdots X_m^{i_m}\},$$

the size of which equals (6). The result now follows from the footprint bound. \square

The above corollary and (4) are clearly related as (6) equals the right side of (4) for $m = 2$. Similarly, (5) equals the assumption in the corollary. Observe, however, that in (4), and in this paper in general, we always assume that the monomial ordering is the lexicographic ordering described in Theorem 9. The master theorem of the present paper is the following result where (7) is the generalization of (4) to more variables and where the mentioned Condition A is the generalization of (5). Recall that $D(i_1, \dots, i_m, r, s_1, \dots, s_m)$ serves as an upper bound on the number of zeros of multiplicity at least r for polynomials with leading monomial being $X_1^{i_1} \cdots X_m^{i_m}$ with respect to the lexicographic ordering. As a consequence the master theorem also can be viewed as a generalization of Corollary 13, when restricted to a lexicographic ordering.

Theorem 14 *Assume that $(i_1, \dots, i_m, r, s_1, \dots, s_m)$ with $m \geq 2$ satisfies Condition A in Definition 15 below. We have*

$$D(i_1, \dots, i_m, r, s_1, \dots, s_m) \leq s_1 \cdots s_m - (s_1 - \frac{i_1}{r}) \cdots (s_m - \frac{i_m}{r}) \quad (7)$$

which is at most equal to $\min\{(i_1 s_2 \cdots s_m + \cdots + s_1 \cdots s_{m-1} i_m)/r, s_1 \cdots s_m\}$.

We postpone the proof of Theorem 14 till the end of the section.

Definition 15 *Let $m \geq 2$. We say that $(i_1, \dots, i_m, r, s_1, \dots, s_m)$ satisfies Condition A if the following hold*

$$\begin{aligned} \text{(A.1)} \quad & 0 \leq i_1 \leq s_1, \dots, 0 \leq i_{m-1} \leq s_{m-1}, 0 \leq i_m < r s_m \\ \text{(A.2)} \quad & s(s_1 - \frac{i_1}{\ell}) \cdots (s_{m-2} - \frac{i_{m-2}}{\ell}) \leq \ell(s_1 - \frac{i_1}{s}) \cdots (s_{m-2} - \frac{i_{m-2}}{s}) \\ & \text{for all } \ell = 2, \dots, r, s = 1, \dots, \ell - 1. \\ \text{(A.3)} \quad & s(s_1 - \frac{i_1}{r}) \cdots (s_{m-1} - \frac{i_{m-1}}{r}) \leq r(s_1 - \frac{i_1}{s}) \cdots (s_{m-1} - \frac{i_{m-1}}{s}) \\ & \text{for all } s = 1, \dots, r - 1. \end{aligned}$$

We note that one could actually replace $\ell = 2, \dots, r$ in (A.2) with the weaker $\ell = 2, \dots, r - 1$ as the case $\ell = r$ follows from (A.3).

Admittedly, the definition of the exponent being small (Condition A) is rather technical. However:

- If (i_1, \dots, i_m) is small then all (i'_1, \dots, i'_m) with $i'_1 \leq i_1, \dots, i'_m \leq i_m$ are also small (Proposition 16). Hence, it is enough to check if (i_1, \dots, i_m) satisfies Condition A.
- Condition A is satisfied when $i_t \leq s_t \min \left\{ \frac{m - \sqrt[r]{r} - 1}{m - \sqrt[r]{r} - \frac{1}{r}}, \frac{m - 2\sqrt[2]{2} - 1}{m - 2\sqrt[2]{2} - \frac{1}{2}} \right\}$, $t = 1, \dots, m - 1$, $i_m < r s_m$ (Theorem 17).
- As already mentioned, Condition A and the master theorem reduces to well-known results when $r = 1$ or when $m = 2$ (see Remark 18 for the details).
- For arbitrary m but $r = 2$ and $s_1 = \cdots = s_m$, Condition A reduces to a simple expression (Proposition 19 and Example 20).

Proposition 16 *If $(i_1, \dots, i_m, r, s_1, \dots, s_m)$ satisfies Condition A then for all i'_1, \dots, i'_m with $0 \leq i'_1 \leq i_1, \dots, 0 \leq i'_m \leq i_m$ also $(i'_1, \dots, i'_m, r, s_1, \dots, s_m)$ satisfies Condition A.*

Proof: It is enough to show that

$$\frac{s_t - \frac{i_t}{s}}{s_t - \frac{i_t}{\ell}} \leq \frac{s_t - \frac{ai_t}{s}}{s_t - \frac{ai_t}{\ell}} \quad (8)$$

holds for all rational numbers a and integers t with $0 < a < 1$ and $1 \leq t \leq m-1$. But (8) is equivalent to $(1-a)(\ell-s) \geq 0$ which is a valid inequality when $\ell > s$. \square

We now give the most important theorem of the paper.

Theorem 17 *If $i_m < rs_m$ and if for $t = 1, \dots, m-1$*

$$i_t \leq s_t \min \left\{ \frac{{}^{m-1}\sqrt{r} - 1}{{}^{m-1}\sqrt{r} - \frac{1}{r}}, \frac{{}^{m-2}\sqrt{2} - 1}{{}^{m-2}\sqrt{2} - \frac{1}{2}} \right\}$$

then $D(i_1, \dots, i_m, r, s_1, \dots, s_m) \leq s_1 \cdots s_m - (s_1 - \frac{i_1}{r}) \cdots (s_m - \frac{i_m}{r})$.

Proof: The idea behind Theorem 17 is to choose i_t , $t = 1, \dots, m-1$ such that

$${}^{m-1}\sqrt{s}(s_t - \frac{i_t}{r}) \leq {}^{m-1}\sqrt{r}(s_t - \frac{i_t}{s}), \quad \text{for } s = 1, \dots, r-1, \quad (9)$$

and such that

$${}^{m-2}\sqrt{s}(s_t - \frac{i_t}{\ell}) \leq {}^{m-2}\sqrt{\ell}(s_t - \frac{i_t}{s}), \quad \text{for } \ell = 2, \dots, r, / s = 1, \dots, \ell-1. \quad (10)$$

The first set of inequalities guarantees (A.3) and the second set guarantees (A.2). Now (9) and (10), respectively, translates to

$$\frac{i_t}{s_t} \leq \frac{\frac{{}^{m-1}\sqrt{r} - 1}{{}^{m-1}\sqrt{r} - \frac{1}{r}} - \frac{{}^{m-1}\sqrt{s}}{s}}{\frac{{}^{m-1}\sqrt{r} - 1}{{}^{m-1}\sqrt{r} - \frac{1}{r}} - \frac{{}^{m-1}\sqrt{s}}{r}}, \quad (11)$$

$$\frac{i_t}{s_t} \leq \frac{\frac{{}^{m-2}\sqrt{\ell} - 1}{{}^{m-2}\sqrt{\ell} - \frac{1}{\ell}} - \frac{{}^{m-2}\sqrt{s}}{s}}{\frac{{}^{m-2}\sqrt{\ell} - 1}{{}^{m-2}\sqrt{\ell} - \frac{1}{\ell}} - \frac{{}^{m-2}\sqrt{s}}{\ell}}, \quad (12)$$

respectively, and then also (A.1) is clearly satisfied. We shall show that the right side of (11) is smallest possible when $s = 1$, in which case it equals $({}^{m-1}\sqrt{r} - 1)/({}^{m-1}\sqrt{r} - 1/r)$. And we shall show that the right side of (12) is smallest possible when $\ell = 2, s = 1$, in which case it equals $({}^{m-2}\sqrt{2} - 1)/({}^{m-2}\sqrt{2} - 1/2)$.

We first consider (11) where we substitute $S = {}^{m-1}\sqrt{s}$ and $R = {}^{m-1}\sqrt{r}$ to obtain

$$\frac{i_t}{s_t} \leq \frac{R^m S^{m-1} - R^{m-1} S^m}{R^m - S^m}.$$

We want to demonstrate that the right side is minimal on $[1, R]$ when $S = 1$. The derivative is

$$\frac{(m-1)R^{2m}S^{m-2} + R^m S^{2m-2} - mR^{2m-1}S^{m-1}}{(R^m - S^m)^2}.$$

Hence, it suffices to show that the numerator is always positive on $]0, R[$. Writing $S = Ra$ with $a \in]0, 1[$ the condition that the numerator should be positive becomes $m-1+a^m-ma > 0$. Plugging in $a = 1$, equality holds. Therefore the result follows from the fact that the derivative of $m-1+a^m-ma$ is negative on $]0, 1[$.

The above proof not only shows that the minimum of the right side of (11) is obtained for $s = 1$. It also applies to demonstrate that the minimum of the right side of (12) is attained in one of the following cases $(\ell = 2, s = 1)$, $(\ell = 3, s = 1)$, $\dots, (\ell = r, s = 1)$. We next substitute $m-2$ with m on the right side of (12)

to obtain $(\ell^{1/m} - 1)/(\ell^{1/m} - 1/\ell)$. We want to show that the minimal value for $\ell \in [2, \infty[$ is attained when $\ell = 2$. The derivative is

$$\frac{-(\ell^{1/m}m - \ell^{(m+1)/m} + \ell^{1/m} - m)}{m(\ell^{(m+1)/m} - 1)^2}$$

where the denominator is always positive and the numerator is positive for $\ell = 0$. The result follows from the fact that

$$\frac{d}{d\ell} (\ell^{1/m}m - \ell^{(m+1)/m} + \ell^{1/m} - m) = \frac{(m+1)(\ell^{-(m-1)/m} - \ell^{1/m})}{m}$$

is negative on $]0, \infty[$. \square

Remark 18 If $r = 1$ then (A.2) and (A.3) do not apply and therefore Condition A reduces to $i_1 \leq s_1, \dots, i_m \leq s_m$. Hence, in this case Theorem 14 in combination with Theorem 9 reduce to Theorem 13.

For $m = 2$ and r arbitrary condition (A.2) does not apply and condition (A.3) simplifies to

$$i_1 \leq \frac{rs}{r+s} s_1$$

for all integers s with $1 \leq s < r$. The minimal upper bound on i_1 is attained for $s = 1$. Hence, in case of two variables Condition A reads $i_1 \leq \frac{r}{r+1} s_1, i_2 < rs_2$. For $m = 2$ and r being arbitrary Theorem 14 therefore equals (4) and (5).

Proposition 19 Assume $r = 2$ and $s_1 = \dots = s_m = q$. Then Condition A simplifies to

$$\sum_{t=1}^{m-1} (-1)^{t+1} \frac{2^{t+1} - 1}{2^t} \sum_{1 \leq j_1 < \dots < j_t \leq m-1} (I_{j_1} \dots I_{j_t}) \leq 1 \quad \text{and} \quad I_m < 2$$

where $I_1 = i_1/q, \dots, I_m = i_m/q$.

Proof: For $r = 2$, the conditions (A.2), (A.3) become

$$(s_1 - \frac{i_1}{2}) \dots (s_{m-1} - \frac{i_{m-1}}{2}) \leq 2(s_1 - i_1) \dots (s_{m-1} - i_{m-1})$$

which is equivalent to

$$\begin{aligned} & \left(1 - \frac{I_1}{2}\right) \dots \left(1 - \frac{I_{m-1}}{2}\right) \leq 2(1 - I_1) \dots (1 - I_{m-1}) \\ & \Updownarrow \\ & 1 + \sum_{t=1}^{m-1} (-1)^t \left(\frac{1}{2}\right)^t \sum_{1 \leq j_1 < \dots < j_t \leq m-1} (I_{j_1} \dots I_{j_t}) \leq \\ & \quad 2 + 2 \sum_{t=1}^{m-1} (-1)^t \sum_{1 \leq j_1 < \dots < j_t \leq m-1} (I_{j_1} \dots I_{j_t}) \\ & \Updownarrow \\ & \sum_{t=1}^{m-1} (-1)^{t+1} \frac{2^{t+1} - 1}{2^t} \sum_{1 \leq j_1 < \dots < j_t \leq m-1} (I_{j_1} \dots I_{j_t}) \leq 1 \end{aligned}$$

and we are through. \square

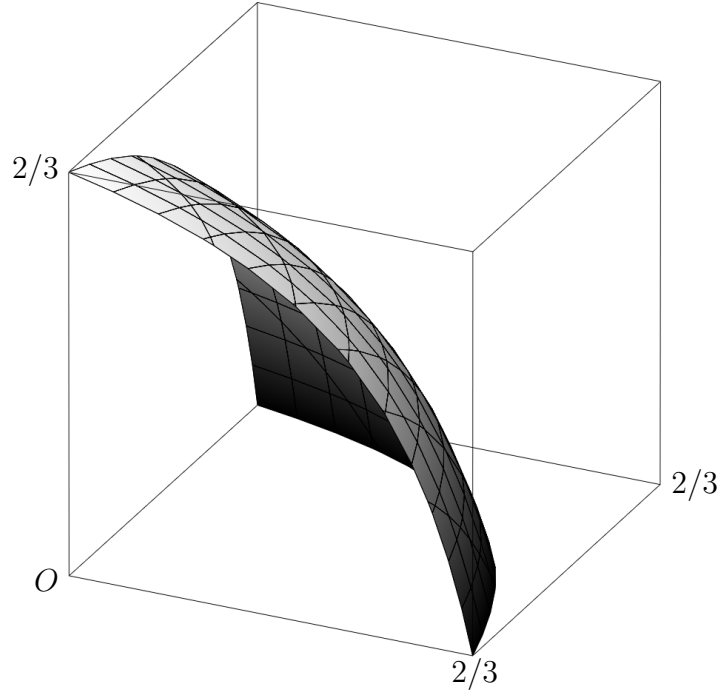


Fig. 1 The surface $\frac{3}{2}(I_1 + I_2 + I_3) - \frac{7}{4}(I_1 I_2 + I_1 I_3 + I_2 I_3) + \frac{15}{8} I_1 I_2 I_3 = 1$

Example 20 Let the notation be as in Proposition 19. For $r = 2$, $m = 3$ and $s_1 = s_2 = s_3 = q$ Condition A reads

$$\frac{3}{2}(I_1 + I_2) - \frac{7}{4} I_1 I_2 \leq 1, \quad I_3 < 2.$$

For $r = 2$, $m = 4$ and $s_1 = s_2 = s_3 = s_4 = q$ Condition A reads

$$\frac{3}{2}(I_1 + I_2 + I_3) - \frac{7}{4}(I_1 I_2 + I_1 I_3 + I_2 I_3) + \frac{15}{8} I_1 I_2 I_3 \leq 1, \quad I_4 < 2.$$

This is illustrated in Figure 1.

From Proposition 19 it is clear that in the case of $r = 2$, for Condition A to hold we must have $i_t \leq \frac{2}{3} s_t$, $t = 1, \dots, m-1$. The general picture for r arbitrary is described in the following proposition.

Proposition 21 Assume that $(i_1, \dots, i_m, r, s_1, \dots, s_m)$ with $m \geq 2$ satisfies Condition A. If $r \geq 2$ then

$$i_1 \leq \frac{r}{r+1} s_1, \dots, i_{m-1} \leq \frac{r}{r+1} s_{m-1}. \quad (13)$$

Proof: Follows from (A.3), the last part of Remark 18, and the fact that

$$s_t - \frac{i_t}{\ell} \geq s_t - \frac{i_t}{s}$$

holds for $t = 1, \dots, m-1$. \square

Proof of Theorem 14: Let $(i_1, \dots, i_m, r, s_1, \dots, s_m)$ with $m \geq 2$ be such that Condition A holds. We give an induction proof that

$$D(i_1, \dots, i_t, l, s_1, \dots, s_t) \leq s_1 \cdots s_t - (s_1 - \frac{i_1}{\ell}) \cdots (s_t - \frac{i_t}{\ell}) \quad (14)$$

for all $1 \leq t < m, 1 \leq \ell \leq r$.

For $t = 1$ the result is clear. Let $1 < t < m$ and assume the result holds when t is substituted with $t - 1$. According to Definition 8 we have

$$\begin{aligned} D(i_1, \dots, i_t, l, s_1, \dots, s_t) = \\ \max_{(u_1, \dots, u_\ell) \in A(i_t, \ell, s_t)} \left\{ (s_t - u_1 - \cdots - u_\ell) D(i_1, \dots, i_{t-1}, \ell, s_1, \dots, s_{t-1}) \right. \\ \left. + u_1 D(i_1, \dots, i_{t-1}, \ell - 1, s_1, \dots, s_{t-1}) + \cdots \right. \\ \left. + u_{\ell-1} D(i_1, \dots, i_{t-1}, 1, s_1, \dots, s_{t-1}) + u_\ell s_1 \cdots s_{t-1} \right\} \end{aligned}$$

where

$$A(i_t, \ell, s_t) = \{(u_1, \dots, u_\ell) \in \mathbb{N}_0^\ell \mid u_1 + \cdots + u_\ell \leq s_t, \quad u_1 + 2u_2 + \cdots + \ell u_\ell \leq i_t\}$$

follows from Definition 8. By the above assumptions this implies that

$$\begin{aligned} D(i_1, \dots, i_t, \ell, s_1, \dots, s_t) \leq \\ \max_{(u_1, \dots, u_\ell) \in B(i_t, \ell, s_t)} \left\{ s_t (s_1 \cdots s_{t-1} - (s_1 - \frac{i_1}{\ell}) \cdots (s_{t-1} - \frac{i_{t-1}}{\ell})) \right. \\ \left. + u_1 ((s_1 - \frac{i_1}{\ell}) \cdots (s_{t-1} - \frac{i_{t-1}}{\ell}) - (s_1 - \frac{i_1}{\ell-1}) \cdots (s_{t-1} - \frac{i_{t-1}}{\ell-1})) \right. \\ \left. + \cdots \right. \\ \left. + u_{\ell-1} ((s_1 - \frac{i_1}{\ell}) \cdots (s_{t-1} - \frac{i_{t-1}}{\ell}) - (s_1 - \frac{i_1}{1}) \cdots (s_{t-1} - \frac{i_{t-1}}{1})) \right. \\ \left. + u_\ell ((s_1 - \frac{i_1}{\ell}) \cdots (s_{t-1} - \frac{i_{t-1}}{\ell})) \right\} \quad (15) \end{aligned}$$

where

$$\begin{aligned} B(i_t, \ell, s_t) = \{(u_1, \dots, u_\ell) \in \mathbb{Q}^\ell \mid 0 \leq u_1, \dots, u_\ell, \quad u_1 + \cdots + u_\ell \leq s_t, \\ \text{and } u_1 + 2u_2 + \cdots + \ell u_\ell \leq i_t\}. \end{aligned}$$

We have $t < m$ and therefore condition (A.2) applies. We note that

$$s(s_1 - \frac{i_1}{\ell}) \cdots (s_{t-1} - \frac{i_{t-1}}{\ell}) \leq \ell(s_1 - \frac{i_1}{s}) \cdots (s_{t-1} - \frac{i_{t-1}}{s})$$

for $s = 1, \dots, \ell - 1$ is equivalent to

$$(\ell - s)(s_1 - \frac{i_1}{\ell}) \cdots (s_{t-1} - \frac{i_{t-1}}{\ell}) \leq \ell(s_1 - \frac{i_1}{\ell - s}) \cdots (s_{t-1} - \frac{i_{t-1}}{\ell - s})$$

for $s = 1, \dots, \ell - 1$ which again is equivalent to

$$\ell((s_1 - \frac{i_1}{\ell}) \cdots (s_{t-1} - \frac{i_{t-1}}{\ell}) - (s_1 - \frac{i_1}{\ell - s}) \cdots (s_{t-1} - \frac{i_{t-1}}{\ell - s})) \leq s(s_1 - \frac{i_1}{\ell}) \cdots (s_{t-1} - \frac{i_{t-1}}{\ell})$$

for $s = 1, \dots, \ell - 1$. Therefore the maximal value of (15) is attained for $u_1 = \cdots = u_{\ell-1} = 0$ and $u_\ell = \frac{i_t}{\ell}$. This concludes the induction proof of (14).

To show (7) we apply similar arguments to the case $t = m$ but use condition (A.3) rather than condition (A.2).

Finally we address the last part of Theorem 14. It is clear that the right side of (7) is smaller than or equal to $s_1 \cdots s_m$. To see that it is also smaller than or equal to

$$\sum_{t=1}^m \left(\prod_{\substack{j=1, \dots, m \\ j \neq t}} s_j \right)^{\frac{i_t}{r}} \quad (16)$$

we start by observing that

$$\left(\prod_{\substack{j=1, \dots, m \\ j \neq t}} s_j \right)^{\frac{i_t}{r}}$$

equals the volume of

$$N(t, \frac{i_t}{r}) = \{(a_1, \dots, a_m) \in \mathbb{R}_0^m \mid 0 \leq a_t < \frac{i_t}{r}, 0 \leq a_j \leq s_j \\ \text{for } j \in \{1, \dots, m\} \setminus \{t\}\}.$$

The sum of volumes of $N(t, \frac{i_t}{r})$, $t = 1, \dots, m$ is larger than or equal to the volume of

$$\cup_{t=1}^m N(t, \frac{i_t}{r}) = \{(a_1, \dots, a_m) \in \mathbb{R}_0^m \mid 0 \leq a_t \leq s_t \text{ for } t = 1, \dots, m \\ \text{and not all } j \text{ satisfy } \frac{i_j}{r} \leq a_j\}$$

which equals the right side of (7). \square

4 Concluding remarks

The results in this paper use the lexicographic ordering. We pose it as a research problem to investigate if some of them hold for arbitrary monomial orderings.

Acknowledgments

This work was supported by the Danish Council for Independent Research (grant no. DFF-4002-00367) and by the Danish National Research Foundation and the National Natural Science Foundation of China (Grant No. 11061130539 – the Danish-Chinese Center for Applications of Algebraic Geometry in Coding Theory and Cryptography).

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